研究ノート

包絡線定理と劣導関数についてのノート

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A note on envelope theorems and subderivatives

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Abstract

In this note, we consider parameterized optimization problems with the abstract choice set. We characterize the value function using subderivatives. From this result, we derive one of Milgrom and Segal's [1] results.

1. Introduction

The envelope theorem states the sufficient condition for the value of a

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parameterized optimization problem to be differentiable with respect to the parameter and provides a formula for its derivative. The envelope theorem is a standard tool in economic analysis (see, for example, Mas-Colell, Whinston, and Green [2]). Various approaches have been studied on this issue in recent years (e.g., Milgrom and Segal [1], Morand, Reffett, and Tarafdar [3], Oyama and Takenawa [4], Marimon and Werner [5]).

Among these, Milgrom and Segal [1] are the major milestone. The traditional envelope theorem requires the choice set to have convexity and topological properties (see, for instance Benveniste and Scheinkman [6]). In contrast, Milgrom and Segal [1] make no such assumptions on the choice set. The reason for using such an abstract choice set is to answer the demands of the models used in recent economics, such as mechanism design.

Milgrom and Segal [1] derive various results, one of which is the characterization of the left and right derivatives of the value function (Milgrom and Segal [1], Theorem 1). It is known that various types of derivatives can be obtained by taking various limits of a function called the difference quotient. (see, for example, Rockafellar and Wets [7]). The right and left derivatives used by Milgrom and Segal are only one of these. They are special cases of the subderivatives.

In this paper, we derive the characterization of the value function using subderivatives. We then derive the Milgrom and Segal's [1] result as a corollary of our result.

The reminder of the paper is organized as follows: In Section 2 we prepare to derive our result. The content of this section relies on Rockafellar and Wets [7]. In Section 3, we present our result and use it to show the Milgrom and Segal's [1] result.

2. Preliminaries

Let $\mathbb{R}=(\infty,\infty)$ be a real line and let $\mathbb{R}^n$ be an n-dimensional Euclidean space. In this paper, we consider expanded real valued functions. That is, the functions take values in the expanded-real-line $\overline{\mathbb{R}}=[\infty,\infty]$ instead of $\mathbb{R}$.

For a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ at a point $\bar{x}\in\overline{\mathbb{R}}$ where $f(\bar{x})$ is finite, that is, $f(\bar{x})\in\mathbb{R}$, the difference quotient function is defined by

$$\Delta_\tau f(\bar{x})(w)=\frac{f(\bar{x}+\tau w)-f(\bar{x})}{\tau}\quad\text{for }\tau\neq 0.$$

By taking limits of different types of the difference quotient, we can get different types of derivatives.

We use the notion $\tau\searrow 0\Leftrightarrow\tau\to 0$ with $\tau>0$. Using this notion, we define subderivatives as follows:

Definition 2.1. For a function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ at a point $\bar{x}\in\mathbb{R}^n$ where $f(\bar{x})$ is finite, the lower subderivative function $d^-f(\bar{x}):\mathbb{R}^n\to\overline{\mathbb{R}}$ is defined by

$$d^-f(\bar{x})(\bar{w})=\liminf_{\tau\searrow 0,\, w\to\bar{w}}\Delta_\tau f(\bar{x})(w),$$

and the upper subderivative function $d^+f(\bar{x}):\mathbb{R}^n\to\overline{\mathbb{R}}$ is defined by

$$d^+f(\bar{x})(\bar{w})=\limsup_{\tau\searrow 0,\, w\to\bar{w}}\Delta_\tau f(\bar{x})(w).$$

The value of $d^-f(\bar{x})(\bar{w})$ (resp. $d^+f(\bar{x})(\bar{w})$) is called the lower (resp. upper) subderivative of f at $\bar{x}$ for $\bar{w}$.

We write $d^-f(\bar{x})$ simply as $df(\bar{x})$ and call it the subderivative function when no confusion will arise. From the definition and the notion, the following relationship holds between the lower subderivative and the upper subderivative function:

$$\begin{aligned}-d(-f)(\bar{x})(\bar{w})&=-\liminf_{\tau\searrow 0,\, w\to\bar{w}}\Delta_\tau f(\bar{x})(w)\\&=\limsup_{\tau\searrow 0,\, w\to\bar{w}}\Delta_\tau f(\bar{x})(w)\\&=d^+f(\bar{x})(w).\end{aligned}$$

We also use the notion $\tau \nearrow 0 \Leftrightarrow \tau \to 0$ with $\tau < 0$. Using this notion, we can introduce the following:

$$-df(\bar{x})(\bar{w}) = \liminf_{\tau \nearrow 0, w \to \bar{w}} \Delta_\tau(-f)(\bar{x})(-w),$$
$$d(-f)(\bar{x})(\bar{w}) = \limsup_{\tau \nearrow 0, w \to \bar{w}} \Delta_\tau(-f)(\bar{x})(-w).$$

In the reminder of this section, we consider functions whose domain is $\mathbb{R}$. For a function $f:\mathbb{R} \to \overline{\mathbb{R}}$ at a point $\bar{x} \in \mathbb{R}$ where $f(\bar{x})$ is finite, the subderivative functions $f(\bar{x})$ are determined from their values at $\bar{w} = \pm 1$. This holds because $df(\bar{x})$ is positive homogeneous (see Rockafellar and Wets [7], Theorem 8.18). There is no need to take the limits $w \to \bar{w}$ in the subderivative functions, so the following holds:

$$\begin{aligned} df(\bar{x})(1) &= \liminf_{\tau \searrow 0} \Delta_\tau f(\bar{x})(1), \\ -d(-f)(\bar{x})(1) &= \limsup_{\tau \searrow 0} \Delta_\tau f(\bar{x})(1), \\ -df(\bar{x})(-1) &= \liminf_{\tau \nearrow 0} \Delta_\tau(-f)(\bar{x})(-1), \qquad (1) \\ d(-f)(\bar{x})(-1) &= \limsup_{\tau \nearrow 0} \Delta_\tau(-f)(\bar{x})(-1). \end{aligned}$$

When $df(\bar{x})(1) = -d(-f)(\bar{x})(1)$ (resp. $-df(\bar{x})(-1) = d(-f)(\bar{x})(-1)$) and the value is finite, we say that f is right (resp. left) differentiable at $\bar{x}$. The value

$$f'_+(\bar{x}) = df(\bar{x})(1) = -d(-f)(\bar{x})(1)$$

is called right derivative of f at $\bar{x}$ and the value

$$f'_-(\bar{x}) = -df(\bar{x})(-1) = d(-f)(\bar{x})(-1)$$

is called left derivative of f at $\bar{x}$. If $f'_+(\bar{x}) = f'_-(\bar{x})$, we say that f is differentiable at $\bar{x}$. We write the common value as $f'(\bar{x})$ and call it the derivative.

3. Results

Let X denote the choice set and let Y be the parameter set. Letting $f: X \times Y \to \overline{\mathbb{R}}$ denote the parameterized objective function, the value function

$v : Y \to \overline{\mathbb{R}}$ and the optimal choice set-valued function $M : Y \rightrightarrows X$ are given by

$$v(y) = \sup_{x \in X} f(x,y),$$
$$M(y) = \{x \in X : f(x,y) = v(y)\}. \tag{2}$$

We denote by $d_y f(\bar{x}, \bar{y})$ the subderivative function associated with $f(\bar{x}, \cdot)$ at $\bar{y}$.

Theorem 3.1. Suppose $Y = \mathbb{R}^n$. Take $\bar{y} \in Y$ and $x^* \in M(y)$, and suppose that $f(x^*, \bar{y})$ is finite. Then,

$$\begin{aligned} dv(\bar{y})(\bar{w}) &\geq d_y f(x^*, \bar{y})(\bar{w}), \\ -d(-v)(\bar{y})(\bar{w}) &\geq -d_y(-f)(x^*, \bar{y})(\bar{w}), \\ -dv(\bar{y})(-\bar{w}) &\leq -d_y f(x^*, \bar{y})(-\bar{w}), \\ d(-v)(\bar{y})(-\bar{w}) &\leq d_y(-f)(x^*, \bar{y})(-\bar{w}). \end{aligned} \tag{3}$$

Proof. Using (2), we have that for any $w \in Y$ that converges to $\bar{w}$ and $\tau > 0$,

$$v(\bar{y} + \tau w) \geq f(x^*, \bar{y} + \tau w).$$

Since $f(x^*, y) = v(y)$,

$$v(\bar{y} + \tau w) - v(y) \geq f(x^*, \bar{y} + \tau w) - f(x^*, \bar{y})$$

holds. Dividing both sides by τ,

$$\frac{v(y + \tau w) - v(y)}{\tau} \geq \frac{f(x^*, y + \tau w) - f(x^*, y)}{\tau}. \tag{4}$$

Taking the lower limit as $\tau \searrow 0$ and $w \to \bar{w}$ on both sides of this inequality, we get that

$$dv(y)(\bar{w}) \geq d_y f(x^*, y)(\bar{w}).$$

Taking the upper limit as $\tau \searrow 0$ and $w \to \bar{w}$ on both sides of the inequality (4), we also get that

$$-d(-v)(y)(\bar{w}) \geq -d_y(-f)(x^*, y)(\bar{w}).$$

Using similar arguments, the remaining inequalities can be derived. □

Using our result, we can derive the Milgrom and Segal's result as a corollary.

Corollary 3.2. (Milgrom and Segal [1], Theorem 1). Suppose $Y=[0,1]$. Take $t\in Y=[0,1]$ and $x^*\in M(t)$, and suppose that the partial derivative $\frac{\partial f(x^*,t)}{\partial t}$ of the objective function with respect to the parameter exists. If $t<1$ and v is right differentiable at t, then

$$v'_+(t)\geq\frac{\partial f(x^*,t)}{\partial t}.$$

If $t>0$ and v is left differentiable at t, then

$$v'_-(t)\leq\frac{\partial f(x^*,t)}{\partial t}.$$

If $t\in(0,1)$ and v is differentialble at t, then

$$v'(t)=\frac{\partial f(x^*,t)}{\partial t}.$$

Proof. Since $\frac{\partial f(x^*,t)}{\partial t}$ exists, $f(x^*,t)$ is finite. If $t<1$, take $t'\in(t,1)$ and $\tau=t'-t>0$. Using the first two inequalities in (1) and (3), we have that

$$dv(t)(1)\geq d_Y f(x^*,t)(1),$$
$$-d(-v)(t)(1)\geq -d_Y(-f)(x^*,t)(1).$$

Form the assumptions, the left-hand sides of these inequalities are equal to $v'_+(t)$ and the right-hand sides are equal to $\frac{\partial f(x^*,t)}{\partial t}$. Then, we obtain that

$$v'_+(t)\geq\frac{\partial f(x^*,t)}{\partial t}. \tag{5}$$

If $t>0$, take $t'\in(0,t)$ and $\tau=t'-t<0$. Using the last inequalities in (1) and (3), we have that

$$-dv(t)(-1)\leq -d_Y f(x^*,t)(-1),$$
$$d(-v)(t)(-1)\leq d_Y(-f)(x^*,t)(-1).$$

Using the assumptions, we obtain the

$$v'_-(t)\leq\frac{\partial f(x^*,t)}{\partial t}. \tag{6}$$

If $t \in (0,1)$ and v is differentiable at t, $v'(t) = v'_{+}(t) = v'_{-}(t)$. Using (5) and (6), we have

$$v'(t) = \frac{\partial f(x^*,t)}{\partial t}.$$

□

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